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Structural Analysis

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J.Y. Richard Liew,
N.E. Shanmugam, and
C.H. Yu
Department of Civil Engineering
The National University of
Singapore, Singapore

2.1 Fundamental Principles

Structural analysis is the determination of forces and deformations of the structure due to applied loads. *Structural design* involves the arrangement and proportioning of structures and their components in such a way that the assembled structure is capable of supporting the designed loads within the allowable limit states. An analytical model is an idealization of the actual structure. The structural model should relate the actual behavior to material properties, structural details, and loading and boundary conditions as accurately as is practicable.

All structures that occur in practice are three-dimensional. For building structures that have regular layout and are rectangular in shape, it is possible to idealize them into two-dimensional frames arranged in orthogonal directions. *Joints* in a structure are those points where two or more members are connected. A truss is a structural system consisting of members that are designed to resist only axial forces. Axially loaded members are assumed to be pin-connected at their ends. A structural system in which joints are capable of transferring end moments is called a *frame*. Members in this system are assumed to be capable of resisting bending moment axial force and shear force. A structure is said to be two dimensional or planar if all the members lie in the same plane. Beams are those members that are subjected to bending or flexure. They are usually thought of as being in horizontal positions and loaded with vertical loads. *Ties* are members that are subjected to axial tension only, while struts (columns or posts) are members subjected to axial compression only.

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The geometric stiffness matrix for a three-dimensional space frame element can be given as follows:

$$[k_g] = \begin{bmatrix} a & 0 & 0 & 0 & -d & -e & -a & 0 & 0 & 0 & -n & -o \\ & b & 0 & d & g & k & 0 & -b & 0 & n & -g & k \\ & & c & e & h & g & 0 & 0 & -c & o & -h & -g \\ & & & f & i & l & 0 & -d & -e & -f & -i & -l \\ & & & & j & 0 & d & -g & h & -i & p & -q \\ & & & & & m & e & -k & -g & -l & q & r \\ & & & & & & a & 0 & 0 & 0 & n & o \\ & & & & & & & b & 0 & -n & g & -k \\ & & & & & & & & c & -o & h & g \\ & & & & & & & & & f & i & l \\ & \text{sym.} & & & & & & & & & j & o \\ & & & & & & & & & & & m \end{bmatrix} \tag{2.361}$$

where

$$a = -\frac{f_6+f_{12}}{L^2}, \quad b = \frac{6f_7}{5L}, \quad c = -\frac{f_5+f_{11}}{L^2}, \quad d = \frac{f_5}{L}, \quad e = \frac{f_6}{L}, \quad f = \frac{f_7 J}{AL},$$
$$g = \frac{f_{10}}{L}, \quad h = -\frac{f_7}{10}, \quad i = \frac{f_6+f_{12}}{6}, \quad j = \frac{2f_7 L}{15}, \quad k = -\frac{f_5+f_{11}}{6}, \quad l = \frac{f_{11}}{L},$$
$$m = \frac{f_{12}}{L}, \quad n = -\frac{f_7 L}{30}, \quad o = -\frac{f_{10}}{2}.$$

Further details can be obtained from [67].

2.13 Structural Dynamic

2.13.1 Equation of Motion

The essential physical properties of a linearly elastic structural system subjected to external dynamic loading are its mass, stiffness properties, and energy absorption capability or damping. The principle of dynamic analysis may be illustrated by considering a simple single-story structure as shown in Figure 2.128. The structure is subjected to a time-varying force $f(t)$. k is the spring constant that relates the lateral story deflection x to the story shear force, and the dash pot relates the damping force to the velocity by a damping coefficient c. If the mass, m, is assumed to concentrate at the beam,

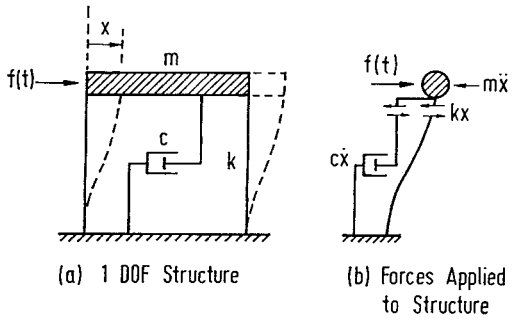


FIGURE 2.128: (a) One DOF structure; (b) forces applied to structures.

the structure becomes a single-degree-of-freedom (SDOF) system. The equation of motion of the

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system may be written as:

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{2.362}$$

Various solutions to Equation 2.362 can give an insight into the behavior of the structure under dynamic situation.

2.13.2 Free Vibration

In this case the system is set to motion and allowed to vibrate in the absence of applied force $f(t)$. Letting $f(t) = 0$, Equation 2.362 becomes

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.363}$$

Dividing Equation 2.363 by the mass m, we have

$$\ddot{x} + 2\xi\omega x + \omega^2 x = 0 \tag{2.364}$$

where

$$2\xi\omega = \frac{c}{m} \quad \text{and} \quad \omega^2 = \frac{k}{m} \tag{2.365}$$

The solution to Equation 2.364 depends on whether the vibration is damped or undamped.

Case 1: Undamped Free Vibration

In this case, $c = 0$, and the solution to the equation of motion may be written as:

$$x = A\sin\omega t + B\cos\omega t \tag{2.366}$$

where $\omega = \sqrt{k/m}$ is the circular frequency. A and B are constants that can be determined by the initial boundary conditions. In the absence of external forces and damping the system will vibrate indefinitely in a repeated cycle of vibration with an amplitude of

$$X = \sqrt{A^2 + B^2} \tag{2.367}$$

and a natural frequency of

$$f = \frac{\omega}{2\pi} \tag{2.368}$$

The corresponding natural period is

$$T = \frac{2\pi}{\omega} = \frac{1}{f} \tag{2.369}$$

The undamped free vibration motion as described by Equation 2.366 is shown in Figure 2.129.

Case 2: Damped Free Vibration

If the system is not subjected to applied force and damping is presented, the corresponding solution becomes

$$x = A\exp(\lambda_1 t) + B\exp(\lambda_2 t) \tag{2.370}$$

where

$$\lambda_1 = \omega\left[-\xi + \sqrt{\xi^2 - 1}\right] \tag{2.371}$$

and

$$\lambda_2 = \omega\left[-\xi - \sqrt{\xi^2 - 1}\right] \tag{2.372}$$

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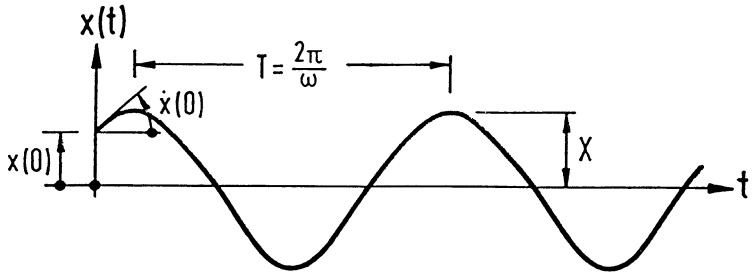


FIGURE 2.129: Response of undamped free vibration.

The solution of Equation 2.370 changes its form with the value of ξ defined as

$$\xi = \frac{c}{2\sqrt{mk}} \tag{2.373}$$

If $\xi^2 < 1$ the equation of motion becomes

$$x = \exp(-\xi\omega t)(A\cos\omega_d t + B\sin\omega_d t) \tag{2.374}$$

where ω_d is the damped angular frequency defined as

$$\omega_d = \sqrt{(1-\xi^2)\omega} \tag{2.375}$$

For most building structures ξ is very small (about 0.01) and therefore $\omega_d \approx \omega$. The system oscillates about the neutral position as the amplitude decays with time t. Figure 2.130 illustrates an example of such motion. The rate of decay is governed by the amount of damping present.

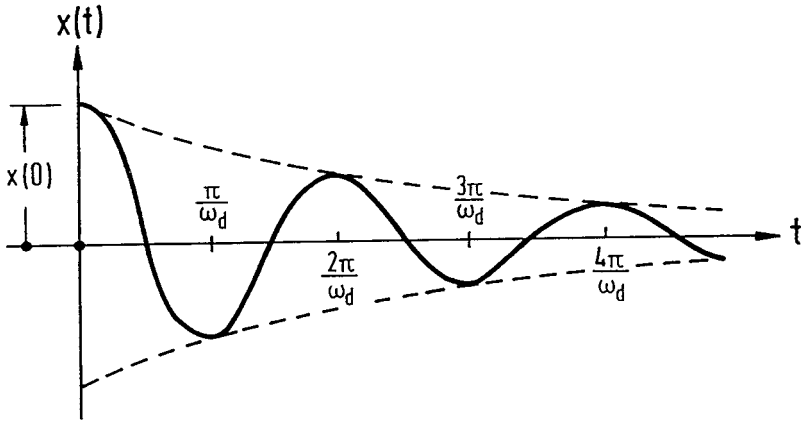


FIGURE 2.130: Response of damped free vibration.

If the damping is large, then oscillation will be prevented. This happens when $\xi^2 > 1$ and the behavior is referred to as *overdamped*. The motion of such behavior is shown in Figure 2.131.

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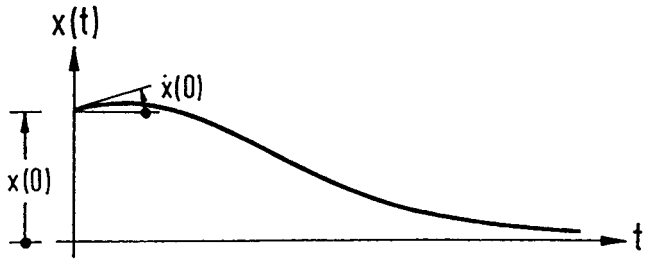


FIGURE 2.131: Response of free vibration with critical damping.

Damping with $\xi^2 = 1$ is called critical damping. This is the case where minimum damping is required to prevent oscillation and the critical damping coefficient is given as

$$c_{cr} = 2\sqrt{km} \tag{2.376}$$

where k and m are the stiffness and mass of the system.

The degree of damping in the structure is often expressed as a proportion of the critical damping value. Referring to Equations 2.373 and 2.376, we have

$$\xi = \frac{c}{c_{cr}} \tag{2.377}$$

ξ is called the critical damping ratio.

2.13.3 Forced Vibration

If a structure is subjected to a sinusoidal motion such as a ground acceleration of $\ddot{x} = F \sin \omega_f t$, it will oscillate and after some time the motion of the structure will reach a steady state. For example, the equation of motion due to the ground acceleration (from Equation 2.364) is

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = -F \sin \omega_f t \tag{2.378}$$

The solution to the above equation consists of two parts; the complimentary solution given by Equation 2.366 and the particular solution. If the system is damped, oscillation corresponding to the complementary solution will decay with time. After some time, the motion will reach a steady state and the system will vibrate at a constant amplitude and frequency. This motion, which is called force vibration, is described by the particular solution expressed as

$$x = C_1 \sin \omega_f t + C_2 \cos \omega_f t \tag{2.379}$$

It can be observed that the steady force vibration occurs at the frequency of the excited force, ω_f, not the natural frequency of the structure, ω.

Substituting Equation 2.379 into Equation 2.378, the displacement amplitude can be shown to be

$$X = -\frac{F}{\omega^2} \frac{1}{\sqrt{\left[\left\{1 - \left(\frac{\omega_f}{\omega}\right)^2\right\}^2 + \left(\frac{2\xi\omega_f}{\omega}\right)^2\right]}} \tag{2.380}$$

The term $-F/\omega^2$ is the static displacement caused by the force due to the inertia force. The ratio of the response amplitude relative to the static displacement $-F/\omega^2$ is called the dynamic displacement

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amplification factor, D, given as

$$D = \frac{1}{\sqrt{\left[\left\{1 - \left(\frac{\omega_f}{\omega}\right)^2\right\}^2 + \left(\frac{2\xi\omega_f}{\omega}\right)^2\right]}} \tag{2.381}$$

The variation of the magnification factor with the frequency ratio ω_f/ω and damping ratio ξ is shown in Figure 2.132.

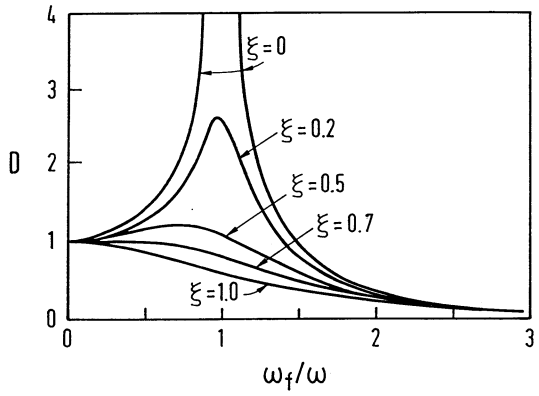


FIGURE 2.132: Variation of dynamic amplification factor with frequency ratio.

When the dynamic force is applied at a frequency much lower than the natural frequency of the system ($\omega_f/\omega \ll 1$), the response is quasi-static. The response is proportional to the stiffness of the structure, and the displacement amplitude is close to the static deflection.

When the force is applied at a frequency much higher than the natural frequency ($\omega_f/\omega \gg 1$), the response is proportional to the mass of the structure. The displacement amplitude is less than the static deflection ($D < 1$).

When the force is applied at a frequency close to the natural frequency, the displacement amplitude increases significantly. The condition at which $\omega_f/\omega = 1$ is known as resonance.

Similarly, the ratio of the acceleration response relative to the ground acceleration may be expressed as

$$D_a = \left|\frac{\ddot{x} + \ddot{x}_g}{\ddot{x}_g}\right| = \sqrt{\frac{1 + \left(\frac{2\xi\omega_f}{\omega}\right)^2}{\left[\left\{1 - \left(\frac{\omega_f}{\omega}\right)^2\right\}^2 + \left(\frac{2\xi\omega_f}{\omega}\right)^2\right]}} \tag{2.382}$$

D_a is called the dynamic acceleration magnification factor.

2.13.4 Response to Suddenly Applied Load

Consider the spring-mass damper system of which a load P_o is applied suddenly. The differential equation is given by

$$M\ddot{x} + c\dot{x} + kx = P_o \tag{2.383}$$

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If the system is started at rest, the equation of motion is

$$x = \frac{P_o}{k}\left[1 - \exp(-\xi\omega t)\left\{\cos\omega_d t + \frac{\xi\omega}{\omega_d}\sin\omega_d t\right\}\right] \tag{2.384}$$

If the system is undamped, then $\xi = 0$ and $\omega_d = \omega$, and we have

$$x = \frac{P_o}{k}\left[1 - \cos\omega_d t\right] \tag{2.385}$$

The maximum displacement is $2(P_o/k)$ corresponding to $\cos\omega_d t = -1$. Since P_o/k is the maximum static displacement, the dynamic amplification factor is equal to 2. The presence of damping would naturally reduce the dynamic amplification factor and the force in the system.

2.13.5 Response to Time-Varying Loads

Some forces and ground motions that are encountered in practice are rather complex in nature. In general, numerical analysis is required to predict the response of such effects, and the finite element method is one of the most common techniques to be employed in solving such problems.

The evaluation of responses due to time-varying loads can be carried out using the Piecewise Exact Method. In using this method, the loading history is divided into small time intervals. Between these points, it is assumed that the slope of the load curve remains constant. The entire load history is represented by a piecewise linear curve, and the error of this approach can be minimize by reducing the length of the time steps. Description of this procedure is given in [21].

Other techniques employed include Fourier analysis of the forcing function followed by solution for Fourier components in the frequency domain. For random forces, random vibration theory and spectrum analysis may be used [25, 61].

2.13.6 Multiple Degree Systems

In multiple degree systems, an independent differential equation of motion can be written for each degree of freedom. The nodal equations of a multiple degree system consisting of n degrees of freedom may be written as

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{F(t)\} \tag{2.386}$$

where $[m]$ is a symmetrical $n \times n$ matrix of mass, $[c]$ is a symmetrical $n \times n$ matrix of damping coefficient, and $\{F(t)\}$ is the force vector which is zero in the case of free vibration.

Consider a system under free vibration without damping. The general solution of Equation 2.386 is assumed in the form of

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} \cos(\omega t - \phi) & 0 & 0 & 0 \\ 0 & \cos(\omega t - \phi) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cos(\omega t - \phi) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix} \tag{2.387}$$

where angular frequency ω and phase angle ϕ are common to all x's. In this assumed solution, ϕ and C_1, C_2, C_n are the constants to be determined from the initial boundary conditions of the motion and ω is a characteristic value (eigenvalue) of the system.

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Substituting Equation 2.387 into Equation 2.386 yields

$$\begin{bmatrix} k_{11}-m_{11}\omega^2 & k_{12}-m_{12}\omega^2 & \cdots & k_{1n}-m_{1n}\omega^2 \\ k_{21}-m_{21}\omega^2 & k_{22}-m_{22}\omega^2 & \cdots & k_{2n}-m_{2n}\omega^2 \\ \vdots & \vdots & \vdots & \vdots \\ k_{n1}-m_{n1}\omega^2 & k_{n2}-m_{n2}\omega^2 & \cdots & k_{nn}-m_{nn}\omega^2 \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix} \cos(\omega t-\phi) = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{2.388}$$

or

$$\Big[[k]-\omega^2[m]\Big]\{C\} = \{0\} \tag{2.389}$$

where $[k]$ and $[m]$ are the $n \times n$ matrices, ω^2 and $\cos(\omega t-\phi)$ are scalars, and $\{C\}$ is the amplitude vector. For non-trivial solutions, $\cos(\omega t-\phi) \neq 0$; thus, solution to Equation 2.389 requires the determinant of $\Big[[k]-\omega^2[m]\Big] = 0$. The expansion of the determinant yields a polynomial of n degree as a function of ω^2, the n roots of which are the eigenvalues $\omega_1, \omega_2, \omega_n$.

If the eigenvalue ω for a normal mode is substituted in Equation 2.389, the amplitude vector $\{C\}$ for that mode can be obtained. $\{C_1\}, \{C_2\}, \{C_3\}, \{C_n\}$ are therefore called the *eigenvectors,* the absolute values of which must be determined through initial boundary conditions. The resulting motion is a sum of n harmonic motions, each governed by the respective natural frequency ω, written as

$$\{x\} = \sum_{i=1}^{n} \{C_i\} \cos(\omega_i t-\phi) \tag{2.390}$$

2.13.7 Distributed Mass Systems

Although many structures may be approximated by lumped mass systems, in practice all structures are distributed mass systems consisting of an infinite number of particles. Consequently, if the motion is repetitive, the structure has an infinite number of natural frequencies and mode shapes. The analysis of a distributed-parameter system is entirely equivalent to that of a discrete system once the mode shapes and frequencies have been determined because in both cases the amplitudes of the modal response components are used as generalized coordinates in defining the response of the structure.

In principle an infinite number of these coordinates is available for a distributed-parameter system, but in practice only a few modes, usually those of lower frequencies, will provide a significant contribution to the overall response. Thus, the problem of a distributed-parameter system can be converted to a discrete system form in which only a limited number of modal coordinates is used to describe the response.

Flexural Vibration of Beams

The motion of the distributed mass system is best illustrated by a classical example of a uniform beam with a span length L and flexural rigidity EI and a self-weight of m per unit length, as shown in Figure 2.133a. The beam is free to vibrate under its self-weight. From Figure 2.133b, dynamic equilibrium of a small beam segment of length dx requires:

$$\frac{\partial V}{\partial x}dx = m dx \frac{\partial^2 y}{\partial t^2} \tag{2.391}$$

in which

$$\frac{\partial^2 y}{\partial x^2} = \frac{M}{EI} \tag{2.392}$$

and

$$V = -\frac{\partial M}{\partial x}, \qquad \frac{\partial V}{\partial x} = -\frac{\partial^2 M}{\partial x^2} \tag{2.393}$$

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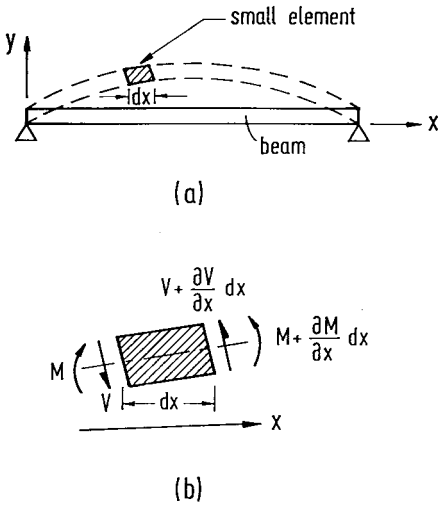


FIGURE 2.133: (a) Beam in flexural vibration; (b) equilibrium of beam segment in vibration.

Substituting these equations into Equation 2.391 leads to the equation of motion of the flexural beam:

$$\frac{\partial^4 y}{\partial x^4} + \frac{m}{EI}\frac{\partial^2 y}{\partial t^2} = 0 \tag{2.394}$$

Equation 2.394 can be solved for beams with given sets of boundary conditions. The solution consists of a family of vibration modes with corresponding natural frequencies. Standard results are available in Table 2.6 to compute the natural frequencies of uniform flexural beams of different supporting conditions. Methods are also available for dynamic analysis of continuous beams [21].

Shear Vibration of Beams

Beams can deform by flexure or shear. Flexural deformation normally dominates the deformation of slender beams. Shear deformation is important for short beams or in higher modes of slender beams. Table 2.7 gives the natural frequencies of uniform beams in shear, neglecting flexural deformation. The natural frequencies of these beams are inversely proportional to the beam length L rather than L^2, and the frequencies increase linearly with the mode number.

Combined Shear and Flexure

The transverse deformation of real beams is the sum of flexure and shear deformations. In general, numerical solutions are required to incorporate both the shear and flexural deformation in the prediction of natural frequency of beams. For beams with comparable shear and flexural deformations, the following simplified formula may be used to estimate the beam's frequency:

$$\frac{1}{f^2} = \frac{1}{f_f^2} + \frac{1}{f_s^2} \tag{2.395}$$

where f is the fundamental frequency of the beam, and f_f and f_s are the fundamental frequencies predicted by the flexure and shear beam theory [50].

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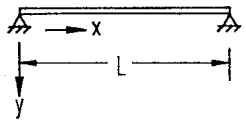
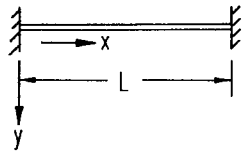
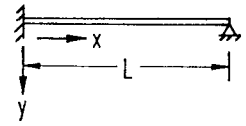
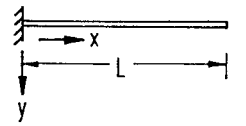
TABLE 2.6 Frequencies and Mode Shapes of Beams in Flexural Vibration

$$f_n = \frac{k_n}{2\pi}\sqrt{\frac{EI}{mL^4}}\ HZ \qquad n = 1, 2, 3...$$

L = Length (m)

EI = Flexural Rigidity (Nm^2)

M = Mass per unit length (kg/m)

Boundary Conditions	K_n; n = 1,2,3	Mode Shape $y_n\left(\frac{x}{L}\right)$	A_n; n = 1,2,3...
Pinned - Pinned 	$(n\pi)^2$	$\sin\frac{n\pi x}{L}$	-
Fixed - Fixed 	22.37 61.67 120.90 199.86 298.55 $(2n+1)\frac{\pi^2}{4}$; n > 5	$\cosh\frac{\sqrt{K_n}x}{L} - \cos\frac{\sqrt{K_n}x}{L}$ $- A_n\left(\sinh\frac{\sqrt{K_n}x}{L} - \sin\frac{\sqrt{K_n}x}{L}\right)$	0.98250 1.00078 0.99997 1.00000 0.99999 1.0; n > 5
Fixed - Pinned 	15.42 49.96 104.25 178.27 272.03 $(4n+1)^2\frac{\pi^2}{4}$; n > 5	$\cosh\frac{\sqrt{K_n}x}{L} - \cos\frac{\sqrt{K_n}x}{L}$ $- A_n\left(\sinh\frac{\sqrt{K_n}x}{L} - \sin\frac{\sqrt{K_n}x}{L}\right)$	1.00078 1.00000 1.0; n > 3
Cantilever 	3.52 22.03 61.69 120.90 199.86 $(2n-1)^2\frac{\pi^2}{4}$; n > 5	$\cosh\frac{\sqrt{K_n}x}{L} - \cos\frac{\sqrt{K_n}x}{L}$ $- A_n\left(\sinh\frac{\sqrt{K_n}x}{L} - \sin\frac{\sqrt{K_n}x}{L}\right)$	0.73410 1.01847 0.99922 1.00003 1.0; n > 4

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TABLE 2.7 Frequencies and Mode Shapes of Beams in Shear Vibration

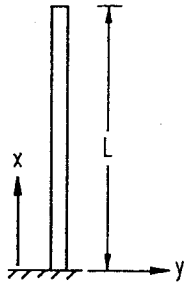
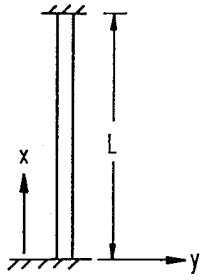
$$f_n = \frac{K_n}{2\pi}\sqrt{\frac{KG}{\rho L^2}} \quad HZ$$

L = Length

K = Shear Coefficient (Cowper, 1966)

G = Shear Modulus = $E/[2(1+\upsilon)]$

ρ = Mass Density

Boundary Condition	K_n; n = 1,2,3...	Mode Shape $y_n\left(\frac{x}{L}\right)$
Fixed - Free 	$n\pi$; n = 1,2,3...	$\cos\frac{n\pi x}{L}$; n = 1,2,3...
Fixed - Fixed 	$n\pi$; n = 1,2,3...	$\sin\frac{n\pi x}{L}$; n = 1,2,3...

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Natural Frequency of Multistory Building Frames

Tall building frames often deform more in the shear mode than in flexure. The fundamental frequencies of many multistory building frameworks can be approximated by [32, 48]

$$f = \alpha \frac{\sqrt{B}}{H} \tag{2.396}$$

where α is approximately equal to $11\sqrt{m}$/s, B is the building width in the direction of vibration, and H is the building height. This empirical formula suggests that a shear beam model with f inversely proportional to H is more appropriate than a flexural beam for predicting natural frequencies of buildings.

2.13.8 Portal Frames

A portal frame consists of a cap beam rigidly connected to two vertical columns. The natural frequencies of portal frames vibrating in the fundamental symmetric and asymmetric modes are shown in Tables 2.8 and 2.9, respectively.

The beams in these frames are assumed to be uniform and sufficiently slender so that shear, axial, and torsional deformations can be neglected. The method of analysis of these frames is given in [68]. The vibration is assumed to be in the plane of the frame, and the results are presented for portal frames with pinned and fixed bases.

If the beam is rigid and the columns are slender and uniform, but not necessarily identical, then the natural fundamental frequency of the frame can be approximated using the following formula [49]:

$$f = \frac{1}{2\pi}\left[\frac{12\sum E_i I_i}{L^3\left(M + 0.37\sum M_i\right)}\right]^{1/2} \text{ Hz} \tag{2.397}$$

where M is the mass of the beam, M_i is the mass of the i-th column, and $E_i I_i$ are the flexural rigidity of the i-th column. The summation refers to the sum of all columns, and i must be greater or equal to 2. Additional results for frames with inclined members are discussed in [11].

2.13.9 Damping

Damping is found to increase with the increasing amplitude of vibration. It arises from the dissipation of energy during vibration. The mechanisms contributing to energy dissipation are material damping, friction at interfaces between components, and energy dissipation due to foundation interacting with soil, among others. Material damping arises from the friction at bolted connections and frictional interaction between structural and non-structural elements such as partitions and cladding.

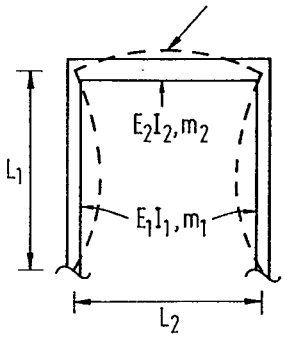
The amount of damping in a building can never be predicted precisely, and design values are generally derived based on dynamic measurements of structures of a corresponding type. Damping can be measured based on the rate of decay of free vibration following an impact; by spectral methods based on analysis of response to wind loading; or by force excitation by mechanical vibrator at varying frequency to establish the shape of the steady state resonance curve. However, these methods may not be easily carried out if several modes of vibration close in frequency are presented.

Table 2.10 gives values of modal damping that are appropriate for use when amplitudes are low. Higher values are appropriate at larger amplitudes where local yielding may develop, e.g., in seismic analysis.

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TABLE 2.8 Fundamental Frequencies of Portal Frames in Asymmetrical Mode of Vibration

First Asymmetric In-Plane Mode



$$f = \frac{\lambda^2}{2\pi L_1^2}\left(\frac{E_1 I_1}{m_1}\right)^{1/2} \text{HZ}$$

E = Modulus of elasticity

I = Area moment of inertia

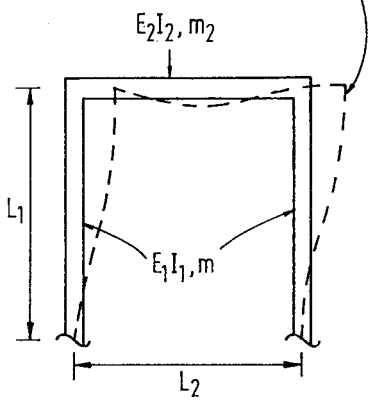
m = mass per unit length

$\frac{m_1}{m_2}$	$\frac{E_1I_1}{E_2I_2}$	λ value									
		Pinned Bases					Clamped Bases				
		L_1/L_2					L_1/L_2				
		0.25	0.75	1.5	3.0	6.0	0.25	0.75	1.5	3.0	6.0
0.25	0.25	0.6964	0.9520	1.1124	1.2583	1.3759	0.9953	1.3617	1.6003	1.8270	2.0193
	0.75	0.6108	0.8961	1.0764	1.2375	1.3649	0.9030	1.2948	1.5544	1.7999	2.0051
	1.5	0.5414	0.8355	1.0315	1.2093	1.3491	0.8448	1.2323	1.5023	1.7649	1.9853
	3.0	0.4695	0.7562	0.9635	1.1610	1.3201	0.7968	1.1648	1.4329	1.7096	1.9504
	6.0	0.4014	0.6663	0.8737	1.0870	1.2702	0.7547	1.1056	1.3573	1.6350	1.8946
0.75	0.25	0.8947	1.1740	1.3168	1.4210	1.4882	1.2873	1.7014	1.9262	2.0994	2.2156
	0.75	0.7867	1.1088	1.2776	1.3998	1.4773	1.1715	1.6242	1.8779	2.0733	2.2026
	1.5	0.6983	1.0368	1.2281	1.3707	1.4617	1.0979	1.5507	1.8218	2.0390	2.1843
	3.0	0.6061	0.9413	1.1516	1.3203	1.4327	1.0373	1.4698	1.7454	1.9838	2.1516
	6.0	0.5186	0.8314	1.0485	1.2414	1.3822	0.9851	1.3981	1.6601	1.9072	2.0983
1.5	0.25	1.0300	1.2964	1.4103	1.4826	1.5243	1.4941	1.9006	2.0860	2.2090	2.2819
	0.75	0.9085	1.2280	1.3707	1.4616	1.5136	1.3652	1.8214	2.0390	2.1842	2.2695
	1.5	0.8079	1.1514	1.3203	1.4326	1.4982	1.2823	1.7444	1.9837	2.1515	2.2521
	3.0	0.7021	1.0482	1.2414	1.3821	1.4694	1.2141	1.6583	1.9070	2.0983	2.2206
	6.0	0.6011	0.9279	1.1335	1.3024	1.4191	1.1570	1.5808	1.8198	2.0234	2.1693
3.0	0.25	1.1597	1.3898	1.4719	1.5189	1.5442	1.7022	2.0612	2.1963	2.2756	2.3190
	0.75	1.0275	1.3202	1.4326	1.4981	1.5336	1.5649	1.9834	2.1515	2.2520	2.3070
	1.5	0.9161	1.2412	1.3821	1.4694	1.5182	1.4752	1.9063	2.0982	2.2206	2.2899
	3.0	0.7977	1.1333	1.3024	1.4191	1.4896	1.4015	1.8185	2.0233	2.1693	2.2595
	6.0	0.6838	1.0058	1.1921	1.3391	1.4395	1.3425	1.7382	1.9366	2.0964	2.2094
6.0	0.25	1.2691	1.4516	1.5083	1.5388	1.5545	1.8889	2.1727	2.2635	2.3228	2.3385
	0.75	1.1304	1.3821	1.4694	1.5181	1.5440	1.7501	2.0980	2.2206	2.2899	2.3268
	1.5	1.0112	1.3023	1.4191	1.4896	1.5287	1.6576	2.0228	2.1693	2.2595	2.3101
	3.0	0.8827	1.1919	1.3391	1.4395	1.5002	1.5817	1.9358	2.0963	2.2095	2.2802
	6.0	0.7578	1.0601	1.2277	1.3595	1.4502	1.5244	1.8550	2.0110	2.1380	2.2309

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TABLE 2.9 Fundamental Frequencies of Portal Frames in Asymmetrical Mode of Vibration.

First Symmetric In-Plane Mode



$$f = \frac{\lambda^2}{2\pi L_1^2}\left(\frac{E_1 I_1}{m_1}\right)^{1/2} \text{HZ}$$

E = Modulus of elasticity

I = Area moment of inertia

m = mass per unit length

$\left(\frac{m_2}{m_1}\right)^{1/4}\left(\frac{E_2I_2}{E_1I_1}\right)^{3/4}$		λ value $\left(\frac{E_1I_1}{E_2I_2}\frac{m_2}{m_1}\right)^{1/4}\frac{L_2}{L_1}$						
		8.0	4.0	2.0	1.0	0.8	0.4	0.2
Pinned Bases	8.0	0.4637	0.8735	1.6676	3.1416	3.5954	3.8355	3.8802
	4.0	0.4958	0.9270	1.7394	3.1416	3.4997	3.7637	3.8390
	2.0	0.5273	0.9911	1.8411	3.1416	3.4003	3.6578	3.7690
	1.0	0.5525	1.0540	1.9633	3.1416	3.3110	3.5275	3.6642
	0.8	0.5589	1.0720	2.0037	3.1416	3.2864	3.4845	3.6240
	0.4	0.5735	1.1173	2.1214	3.1416	3.2259	3.3622	3.4903
	0.2	0.5819	1.1466	2.2150	3.1416	3.1877	3.2706	3.3663
Clamped Bases	8.0	0.4767	0.8941	1.6973	3.2408	3.9269	4.6167	4.6745
	4.0	0.5093	0.9532	1.7847	3.3166	3.9268	4.5321	4.6260
	2.0	0.5388	1.0185	1.9008	3.4258	3.9268	4.4138	4.5454
	1.0	0.5606	1.0773	2.0295	3.5564	3.9267	4.2779	4.4293
	0.8	0.5659	1.0932	2.0696	3.5988	3.9267	4.2351	4.3861
	0.4	0.5776	1.1316	2.1790	3.7176	3.9267	4.1186	4.2481
	0.2	0.5842	1.1551	2.2575	3.8052	3.9266	4.0361	4.1276

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TABLE 2.10 Typical Structural Damping Values

Structural type	Damping value, ξ (%)
Unclad welded steel structures	0.3
Unclad bolted steel structures	0.5
Floor, composite and non-composite	1.5-3.0
Clad buildings subjected to sidesway	1

2.13.10 Numerical Analysis

Many less complex dynamic problems can be solved without much difficulty by hand methods. For more complex problems, such as determination of natural frequencies of complex structures, calculation of response due to time-varying loads, and response spectrum analysis to determine seismic forces, may require numerical analysis. The finite element method has been shown to be a versatile technique for this purpose.

The global equations of an undamped force-vibration motion, in matrix form, may be written as

$$[M]\{\ddot{x}\} + [K]\{\dot{x}\} = \{F(t)\} \tag{2.398}$$

where

$$[K] = \sum_{i=1}^{n} [k_i] \qquad [M] = \sum_{i=1}^{n} [m_i] \qquad [F] = \sum_{i=1}^{n} [f_i] \tag{2.399}$$

are the global stiffness, mass, and force matrices, respectively. $[k_i]$, $[m_i]$, and $\{f_i\}$ are the stiffness, mass, and force of the i^{th} element, respectively. The elements are assembled using the direct stiffness method to obtain the global equations such that intermediate continuity of displacements is satisfied at common nodes and, in addition, interelement continuity of acceleration is also satisfied.

Equation 2.398 is the matrix equation discretized in space. To obtain solution of the equation, discretization in time is also necessary. The general method used is called direct integration. There are two methods for direct integration: *implicit* or *explicit*. The first, and simplest, is an explicit method known as the central difference method [9]. The second, more sophisticated but more versatile, is an implicit method known as the Newmark method [44]. Other integration methods are also available in [7].

The natural frequencies are determined by solving Equation 2.398 in the absence of force $F(t)$ as

$$[M]\{\ddot{x}\} + [K]\{x\} = 0 \tag{2.400}$$

The standard solution for $x(t)$ is given by the harmonic equation in time

$$\{x(t)\} = \{X\}e^{i\omega t} \tag{2.401}$$

where $\{X\}$ is the part of the nodal displacement matrix called natural modes, which are assumed to be independent of time, i is the imaginary number, and ω is the natural frequency.

Differentiating Equation 2.401 twice with respect to time, we have

$$\ddot{x}(t) = \{X\}\left(-\omega^2\right)e^{i\omega t} \tag{2.402}$$

Substituting of Equations 2.401 and 2.402 into Equation 2.400 yields

$$e^{i\omega t}\left([K] - \omega^2[M]\right)\{X\} = 0 \tag{2.403}$$

Since $e^{i\omega t}$ is not zero, we obtain

$$\left([K] - \omega^2[M]\right)\{X\} = 0 \tag{2.404}$$

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Equation 2.404 is a set of linear homogeneous equations in terms of displacement mode $\{X\}$. It has a non-trivial solution if the determinant of the coefficient matrix $\{X\}$ is non-zero; that is

$$[K] - \omega^2[M] = 0 \tag{2.405}$$

In general, Equation 2.405 is a set of n algebraic equations, where n is the number of degrees of freedom associated with the problem.

2.14 Defining Terms

Arch: Principal load-carrying member curved in elevation; resistance to applied loading developed by axial thrust and bending.

Beam: A straight or curved structural member, primarily supporting loads applied at right angles to the longitudinal axis.

Bending moment: Bending moment due to a force or a system of forces at a cross-section is computed as the algebraic sum of all moments to one side of the section.

Built-in beam: A beam restrained at its ends against vertical movement and rotation.

Cables: Flexible structures with no moment-carrying capacity.

Cantilever: A beam restrained against movement and rotation at one end and free to deflect at the other end.

Continuous beam: A beam that extends over several supports.

Deflection: Movement of a structure or parts of a structure under applied loads.

Element: Part of a cross-section forming a distinct part of the whole.

Grillage: Structures in which the members all lie in one plane with loads being applied in the direction normal to this plane.

Hogging moment: Bending moment causing upward deflection in a beam.

Influence line: An influence line indicates the effect at a given section of a unit load placed at any point on the structure.

Member: Any individual component of a structural frame.

Moment of inertia: The second moment of area of a section about the elastic neutral axis.

Plastic analysis: Analysis assuming redistribution of moments within the structure in a continuous construction.

Plastic hinge: Position at which a member has developed its plastic moment of resistance.

Plastic moment: Moment capacity allowing for redistribution of stress within a cross-section.

Plastic section: A cross-section that can develop a plastic hinge with sufficient rotational capacity to allow redistribution of bending moments within the section.

Portal frame: A single-story continuous plane frame deriving its strength from bending resistance and arch action.

Reaction: The load carried by each support.

Rigid frame: An indeterminate plane frame consisting of members with fixed end connections.

Sagging moment: An applied bending moment causing a sagging deflection in the beam.

Second-order analysis: Analysis considering the equilibrium formulated based on deformed structural geometry.

Semi-rigid connection: A connection that possesses a moment capacity intermediate between the simple and rigid connection.

Shear force: An internal force acting normal to the longitudinal axis; given by the algebraic sum

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of all forces to one side of the section chosen.

Simple beam: A beam restrained at its end only against vertical movement.

Space frame: A three-dimensional structure.

Span: The distance between the supports of a beam or a truss.

Static load: A noncyclic load that produces no dynamic effects.

Statically determinate structure: A structure in which support reactions may be found from the equations of equilibrium.

Statically indeterminate structure: A structure in which equations of equilibrium are not sufficient to determine the reactions.

Thin plate: A flat surface structure in which the thickness is small compared to the other dimensions.

Thin shell: A curved surface structure with a thickness relatively small compared to its other dimensions.

Truss: A coplanar system of structural members joined at their ends to form a stable framework.

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Further Reading

The *Structural Engineering Handbook* by E. H. Gaylord and C. N. Gaylord provides a reference work on structural engineering and deals with planning, design, and construction of a variety of engineering structures.

The *Finite Element Handbook* by H. Kardestuncer and D. H. Norrie presents the underlying mathematical principles, the fundamental formulations, and both commonly used and specialized applications of the finite element method.

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